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# Duality transformations of string effective actions

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We show that the duality transformation relating the two versions of  $d=10$ ,  $N=1$  supergravity can be extended iteratively to the string effective actions and supersymmetry transformation rules. We present this extended duality transformation explicitly for the quartic effective actions.

In a recent paper [1] we constructed the quartic effective action for the heterotic string in the dual formulation of  $d=10$ ,  $N=1$  supergravity [2]. The construction of this supersymmetric action followed the lines set out in ref. [3], where a similar result was obtained for the two-index formulation of the supergravity theory. In this note, we will show that the effective actions in these two formulations are related by an iterative generalisation of the duality transformation.

Let us first sketch the structure of the standard duality transformation [4]. The  $d=10$ ,  $N=1$  supergravity theory contains, in its usual formulation, a two-index antisymmetric tensor gauge field  $B_{\mu\nu}$ , which occurs everywhere in the form  $\partial_{[\mu}B_{\nu\rho]}$ . We replace  $\partial_{[\mu}B_{\nu\rho]}$  by an unconstrained antisymmetric tensor field  $t_{\mu\nu\rho}$  everywhere in the action and transformation rules, and add a term with a six-index gauge field to the action:

$$\mathcal{L} = \mathcal{L}(t) - i\epsilon^{\alpha_1 \dots \alpha_6 \lambda \mu \nu \rho} A_{\alpha_1 \dots \alpha_6} \partial_\lambda t_{\mu \nu \rho}. \quad (1)$$

The original theory can be recovered by solving the equation of motion of  $A_{\alpha_1 \dots \alpha_6}$ . On the other hand, one can solve the equation of motion of  $t_{\mu \nu \rho}$  to obtain the dual version of the theory. Supersymmetry is restored by first assigning to  $t$  the transformation rule of  $\partial B$ . Then the action (1) is invariant except for variations proportional to  $\nabla_{[\lambda} t_{\mu \nu \rho]}$ . These are cancelled by choosing the transformation of  $A_{\alpha_1 \dots \alpha_6}$  appropriately.

In the present case the lagrangian  $\mathcal{L}(t)$  takes on the form

$$\mathcal{L}(t) = \mathcal{L}_0(t) + \sum_{n=1} \mathcal{L}_n(t), \quad \mathcal{L}_0(t) = A t_{\mu \nu \rho} t^{\mu \nu \rho} + C^{\mu \nu \rho} t_{\mu \nu \rho} + D, \quad (2)$$

where  $\mathcal{L}_n$  is of  $O(\gamma^n)$ ,  $\gamma$  being a suitable expansion parameter (a coupling constant, the inverse string tension).  $\mathcal{L}_0$  represents the pure supergravity theory without matter coupling.  $A$ ,  $C_{\mu \nu \rho}$  and  $D$  are independent of  $t$ . The equation of motion of  $t$  is given by

$$t_{\mu \nu \rho} = h_{\mu \nu \rho} - \frac{1}{2} A^{-1} C_{\mu \nu \rho} - \frac{1}{2} A^{-1} \sum_{n=1} \delta \mathcal{L}_n / \delta t^{\mu \nu \rho}, \quad (3)$$

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with

$$h_{\mu\nu\rho} \equiv -\frac{1}{2} \mathbf{1} A^{-1} \epsilon^{\alpha_1 \dots \alpha_7}{}_{\mu\nu\rho} R(A)_{\alpha_1 \dots \alpha_7} . \quad (4)$$

The terms of higher order in  $\gamma$  in (3) are not independent of and not even algebraic in  $t$  in the application we will consider. Nevertheless, by using (3) iteratively we can obtain in principle the duality transformation to any given order in  $\gamma$ . To show this explicitly for the first few orders in  $\gamma$  it is convenient to rewrite the action (1) as follows:

$$\mathcal{L} = -A(h_{\mu\nu\rho} - \frac{1}{2} A^{-1} C_{\mu\nu\rho})^2 + D + A(t_{\mu\nu\rho} - h_{\mu\nu\rho} + \frac{1}{2} A^{-1} C_{\mu\nu\rho})^2 + \sum_{n=1} \mathcal{L}_n \quad (5)$$

In lowest order all  $\mathcal{L}_n = 0$  for  $n \geq 1$ . In that case we can immediately substitute the solution  $t = t_0 \equiv h - \frac{1}{2} C/A$  of the  $t$  equation of motion in (5), the result being

$$\mathcal{L} = \mathcal{L}_0 = -At_0^2 + D = -Ah_{\mu\nu\rho} h^{\mu\nu\rho} + C^{\mu\nu\rho} h_{\mu\nu\rho} + D - \frac{1}{4} A^{-1} C^{\mu\nu\rho} C_{\mu\nu\rho} . \quad (6)$$

Note that the leading term has a minus-sign compared to (2), and that terms quadratic in  $C$  appear. These are usually four-fermion interactions. Otherwise the action keeps the same functional form, with  $t$  replaced by  $h$ .

Now consider the duality transformation at  $O(\gamma)$ , i.e., we let  $\mathcal{L}_1 \neq 0$ . Note that (5) is quadratic in  $t_1 \equiv -\frac{1}{2} A^{-1} \delta \mathcal{L}_1 / \delta t|_{t_0}$ , the  $O(\gamma)$  contribution to  $t$  in the solution of (3). Therefore the action to  $O(\gamma)$  reads

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 = -At_0^2 + D + \mathcal{L}_1|_{t_0} . \quad (7)$$

Of course we must also replace  $t$  by  $t_0 + t_1$  in the supersymmetry transformation rules.

At  $O(\gamma^2)$  we obtain the following action after the duality transformation:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 = -At_0^2 + D + \mathcal{L}_1|_{t_0+t_1} + \mathcal{L}_2|_{t_0} + At_1^2 = -At_0^2 + D + (\mathcal{L}_1 + \mathcal{L}_2)|_{t_0} - At_1^2 . \quad (8)$$

Hence the iterative procedure generates terms of  $O(\gamma^2)$  besides those already present in  $\mathcal{L}_2$ . The supersymmetry transformation rules also depend on  $t_2$ , the  $O(\gamma^2)$  contribution to the solution for  $t$ . It is given by

$$t_2 = -\frac{1}{2} A^{-1} (\delta \mathcal{L}_1 / \delta t|_{t_0+t_1} + \delta \mathcal{L}_2 / \delta t|_{t_0}) - t_1 . \quad (9)$$

We finally consider the duality transformation at  $O(\gamma^3)$ . The action reads

$$\begin{aligned} \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 &= -At_0^2 + D + \mathcal{L}_1|_{t_0+t_1+t_2} + \mathcal{L}_2|_{t_0+t_1} + \mathcal{L}_3|_{t_0} + A(t_1^2 + 2t_1 t_2) \\ &= -At_0^2 + D + (\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3)|_{t_0} - A(t_1^2 + 2t_1 t_2) - \frac{1}{2} (\delta^2 \mathcal{L}_1 / \delta t^2)|_{t_0} t_1 t_1 . \end{aligned} \quad (10)$$

We refrain from giving explicitly  $t_3$ , the  $O(\gamma^3)$  contribution to  $t$ , which is needed for the supersymmetry transformation rules.

Note that in  $O(\gamma^n)$  the duality transformation for the action does not depend on  $t_n$ . However,  $t_n$  is needed for the supersymmetry transformation rules.

Let us now apply the above formalism to the specific case of the  $d=10$  effective action. In the string effective action there is one feature that we can treat immediately to all orders in  $\alpha$  and  $\beta$ <sup>#1</sup>. The field strength  $\partial_{[\mu} B_{\nu\rho]}$  occurs everywhere in the combination

$$H_{\mu\nu\rho} = \delta_{[\mu} B_{\nu\rho]} - X_{\mu\nu\rho} , \quad (11)$$

where  $X_{\mu\nu\rho}$  is the combination of Yang–Mills (proportional to  $\beta$ ) and Lorentz ( $\alpha$ ) Chern–Simons forms required for anomaly cancellations [5]. It is then convenient to rewrite (1) in the form

$$\mathcal{L} = \mathcal{L}(t') + \mathbf{1} \epsilon^{\alpha_1 \dots \alpha_7}{}_{\mu\nu\rho} R(A)_{\alpha_1 \dots \alpha_7} t'_{\mu\nu\rho} - \mathbf{1} \epsilon^{\alpha_1 \dots \alpha_6}{}_{\lambda\mu\nu\rho} A_{\alpha_1 \dots \alpha_6} \partial_\lambda X_{\mu\nu\rho} , \quad (12)$$

<sup>#1</sup> The generic parameter  $\gamma$  considered thus far is now replaced by  $\alpha$ , the inverse string tension, and  $\beta$ , the inverse Yang–Mills coupling squared. The contributions  $\mathcal{L}_n$  to the action contain  $\alpha^k \beta^l$ , with  $k+l=n$ .

where  $t' \equiv t - X$ . If we then repeat the above duality transformations using  $t'$  we see that we will get the same actions (5), (7), (8) and (10), but now containing also the last term in (12). Therefore the action in the dual formulation does not contain the Chern–Simons terms in a generalized field strength, but their role is taken over by the last term in (12). This leads to the same anomaly cancellations as in the two-index formulation of the theory [6,7].

Using a few specific aspects of the effective action in the two-index formulation, we can already anticipate most of the characteristic features of the dual version. These aspects are:

- (a)  $C_{\mu\nu\rho}$  [see eq. (2)] is bilinear in fermions.
- (b)  $t_1 \equiv t_{1,f} + t_{1,em}$ , where  $t_{1,f}$  is bilinear in fermions, and  $t_{1,em}$  is proportional to  $O(\alpha^0)$  equations of motion of the supergravity fields.
- (c) The bosonic part of  $\mathcal{L}_2$  vanishes.

Using (a), (b) and (c), and the general formulae given above, one finds that the  $\mathcal{L}_n$  in the dual formulation have essentially the same form as in the two-index version. However, besides terms quartic in fermions, there are the following differences:

- ( $n=0$ ) In  $\mathcal{L}_0$  the sign of the  $h^2$ -term has changed.
- ( $n=1$ )  $\mathcal{L}_1$  now contains the last term in (12).
- ( $n=2$ )  $\mathcal{L}_2$  contains additional terms of the form  $t_{1,f}t_{1,em}$ . These can be eliminated by a field redefinition, proportional to  $t_{1,f}$ , which however modifies  $\mathcal{L}_3$  by terms proportional to  $t_{1,f}$ .
- ( $n=3$ ) Besides the terms coming from the redefinition at  $n=2$ , all modifications to  $\mathcal{L}_3$  can be pushed to higher orders by field redefinitions.

So far the only modifications to the bosonic terms in the action are the sign of  $\mathcal{L}_0$  and the new term in  $\mathcal{L}_1$ . We expect further bosonic modifications which cannot be eliminated by field redefinitions at  $O(\alpha^6)$ . This is because  $t_1$  and  $t_2$  are either fermionic, or proportional to field equations, but  $t_3$  is not. The first new term is therefore proportional to  $t_3^2$ , where  $t_3$  is the contribution of  $O(\alpha^3)$  to the  $t$  equation of motion.

Let us now present some of the details of these duality transformations. We start with the usual [7,8] duality transformation in  $d=10$ ,  $N=1$  supergravity coupled to Yang–Mills theory [9]. This will set the stage for the subsequent calculation of the duality transformation for the effective action. In a suitable basis, the action reads (we use throughout this paper the notation of ref. [3])

$$\begin{aligned} \mathcal{L} = & \mathcal{L}_0(R) + \mathcal{L}_\beta(F^2) + \mathcal{L}_{\beta^2}, \\ \mathcal{L}_0(R) = & e\phi^{-3} \left[ -\frac{1}{2}R(\omega(e)) - \frac{3}{4}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{9}{2}(\phi^{-1}\partial_\mu\phi)^2 - \frac{1}{2}\bar{\psi}_\mu\Gamma^{\mu\nu\rho}\mathcal{D}_\nu(\omega(e))\psi_\rho \right. \\ & + 2\sqrt{2}\bar{\lambda}\Gamma^{\mu\nu}\mathcal{D}_\mu(\omega(e))\psi_\nu + 4\bar{\lambda}\mathcal{D}(\omega(e))\lambda + 3\sqrt{2}\bar{\psi}_\mu\Gamma^\nu\Gamma^\mu\lambda(\phi^{-1}\partial_\nu\phi) - \frac{3}{2}\bar{\psi}_\mu\Gamma^\mu\psi_\nu(\phi^{-1}\delta^\nu\phi) \\ & + \frac{1}{16}\sqrt{2}H^{\rho\sigma\tau}(\bar{\psi}_\mu\Gamma^{\mu\rho}\Gamma_{\rho\sigma\tau}\Gamma^\nu\psi_\nu + 4\sqrt{2}\bar{\psi}_\mu\Gamma^\mu\rho_{\sigma\tau}\lambda - 8\bar{\lambda}\Gamma_{\rho\sigma\tau}\lambda) \\ & \left. + \frac{1}{96}\bar{\psi}^\mu\Gamma^{abc}\psi_\mu(\bar{\lambda}\Gamma_{abc}\lambda + \frac{1}{2}\sqrt{2}\bar{\lambda}\Gamma_{abc}\Gamma^\cdot\psi - \frac{1}{4}\bar{\psi}^\nu\Gamma_{abc}\psi_\nu - \frac{1}{8}\bar{\psi}^\cdot\Gamma\Gamma_{abc}\Gamma^\cdot\psi) \right], \end{aligned} \quad (13)$$

$$\begin{aligned} \mathcal{L}_\beta(F^2) = & e\phi^{-3}\beta \operatorname{tr} \left[ -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\bar{\chi}\mathcal{D}(\omega(e, \psi), A)\chi - \frac{1}{8}\bar{\chi}\Gamma^\mu\Gamma^{ab}(F_{ab} + \hat{F}_{ab})(\psi_\mu + \frac{1}{3}\sqrt{2}\Gamma_\mu\lambda) \right. \\ & \left. + \frac{1}{16}\sqrt{2}\bar{\chi}\Gamma^{abc}\chi\hat{H}_{abc} + \frac{1}{96}\bar{\chi}\Gamma^{abc}\chi\bar{\lambda}\Gamma_{abc}\lambda - [1/(16 \times 24)]\sqrt{2}\bar{\chi}\Gamma^{abc}\chi\bar{\psi}_\mu(4\Gamma_{abc}\Gamma^\mu + 3\Gamma^\mu\Gamma_{abc})\lambda \right], \end{aligned} \quad (14)$$

$$\mathcal{L}_{\beta^2} = -[1/(16 \times 24)]\beta e\phi^{-3} \operatorname{tr}(\bar{\chi}\Gamma^{abc}\chi) \operatorname{tr}(\bar{\chi}\Gamma_{abc}\chi). \quad (15)$$

This action is invariant under the following supersymmetry transformations:

$$\begin{aligned} \delta_0 e_\mu{}^a = & \frac{1}{2}\bar{\epsilon}\Gamma^a\psi_\mu, \\ \delta_0 \psi_\mu = & (\partial_\mu - \frac{1}{4}\Omega_{\mu+}^{ab}\Gamma_{ab})\epsilon + \frac{1}{2}\sqrt{2}(\epsilon\bar{\psi}_\mu\lambda - \psi_\mu\bar{\epsilon}\lambda + \Gamma^a\lambda\bar{\psi}_\mu\Gamma_a\epsilon), \\ \delta_0 B_{\mu\nu} = & \frac{1}{2}\sqrt{2}\bar{\epsilon}\Gamma_{[\mu}\psi_{\nu]}, \\ \delta_0 \lambda = & -\frac{3}{8}\sqrt{2}\phi^{-1}\mathcal{D}\phi\epsilon + \frac{1}{8}\Gamma^{abc}\epsilon(\hat{H}_{abc} - \frac{1}{24}\sqrt{2}\bar{\lambda}\Gamma_{abc}\lambda), \end{aligned} \quad (16)$$

$$\begin{aligned}
\phi^{-1} \delta_0 \phi &= -\frac{1}{3} \sqrt{2} \bar{\epsilon} \lambda, \\
\delta_0 A_\mu &= \frac{1}{2} \bar{\epsilon} \Gamma_\mu \chi, \\
\delta_0 \chi &= -\frac{1}{4} \Gamma^{ab} \epsilon \hat{F}_{ab} + \frac{1}{2} \sqrt{2} (\epsilon \bar{\chi} \lambda - \chi \bar{\epsilon} \lambda + \Gamma^a \lambda \bar{\chi} \Gamma_a \epsilon), \\
\delta_\beta \psi_\mu &= \frac{1}{192} \beta \Gamma^{abc} \Gamma_\mu \epsilon \operatorname{tr}(\bar{\chi} \Gamma_{abc} \chi), \\
\delta_\beta B_{\mu\nu} &= -\beta \sqrt{2} \operatorname{tr}(A_{[\mu} \delta_0 A_{\nu]}), \\
\delta_\beta \lambda &= \frac{1}{384} \beta \sqrt{2} \Gamma^{abc} \epsilon \operatorname{tr}(\bar{\chi} \Gamma_{abc} \chi),
\end{aligned} \tag{16 cont'd}$$

where  $\delta_{\beta^n}$  indicates contributions of  $O(\beta^n)$  in the Yang–Mills coupling constant  $\beta$ . Derivatives  $\mathcal{D}$  are covariant with respect to Lorentz and Yang–Mills gauge transformations, while  $D$  is also supercovariant. On curvatures we indicate supercovariance with a hat. The field-strength  $H_{\mu\nu\lambda}$  of  $B_{\mu\nu}$  is given by (11), with

$$X_{\mu\nu\rho} = \beta \sqrt{2} \operatorname{tr}(A_{[\mu} \partial_\nu A_{\rho]} - \frac{2}{3} A_{[\mu} A_\nu A_{\rho]}). \tag{18}$$

The combination  $\Omega_{\mu\pm}^{ab}$  is given by

$$\Omega_{\mu\pm}^{ab} = \omega_\mu^{ab}(e, \psi) \pm \frac{3}{2} \sqrt{2} \hat{H}_{\mu ab}, \tag{19}$$

where  $\omega(e, \psi)$  is the solution of  $D_{[\mu}(\omega) e_{\nu]}^a = 0$ .

In performing the duality transformation we should use (12), with the Yang–Mills Chern–Simons term  $X$  given by (18). Let us drop the primes on  $t$  from here on. The contributions to the solution of the  $t$  equation of motion are given by

$$t_{0\mu\nu\rho} = \frac{3}{2} i e^{-1} \phi^3 \epsilon^{\alpha_1 \alpha_7}{}_{\mu\nu\rho} R(A)_{\alpha_1 \alpha_7} + \frac{1}{24} \sqrt{2} (\bar{\psi}_\sigma \Gamma^{[\sigma} \Gamma_{\mu\nu\rho} \Gamma^{\tau]} \psi_\tau + 4 \sqrt{2} \bar{\psi}_\sigma \Gamma^\sigma{}_{\mu\nu\rho} \lambda - 8 \bar{\lambda} \Gamma_{\mu\nu\rho} \lambda), \tag{20}$$

$$t_{\beta\mu\nu\rho} \equiv -\frac{1}{2} A^{-1} \delta \mathcal{L}_\beta / \delta t^{\mu\nu\rho} = \frac{1}{24} \beta \sqrt{2} \operatorname{tr}(\bar{\chi} \Gamma_{\mu\nu\rho} \chi). \tag{21}$$

In this case the action  $\mathcal{L}_\beta$  is linear in, and  $\mathcal{L}_{\beta^2}$  independent of  $H$ , and therefore the *exact* result of the duality transformation is given by (8), if we include the last term of (12). If we use the combinations

$$\begin{aligned}
H_{abc} &= \frac{3}{2} i e^{-1} \phi^3 \epsilon^{\alpha_1 \alpha_7}{}_{abc} R(A)_{\alpha_1 \alpha_7}, \\
\hat{H}_{abc} &= \frac{3}{2} i e^{-1} \phi^3 \epsilon^{\alpha_1 \alpha_7}{}_{abc} \hat{R}(A)_{\alpha_1 \alpha_7} - \frac{1}{3} \sqrt{2} \bar{\lambda} \Gamma_{abc} \lambda,
\end{aligned} \tag{22}$$

the dual version of  $d=10$ ,  $N=1$  supergravity coupled to Yang–Mills can be written in the following form:

$$\begin{aligned}
\mathcal{L}_0(R) &= e \phi^{-3} [-\frac{1}{2} R(\omega(e)) + \frac{3}{4} H_{abc} H^{abc} + \frac{9}{2} (\phi^{-1} \partial_\mu \phi)^2 - \frac{1}{2} \bar{\psi}_\mu \Gamma^{\mu\nu\rho} \mathcal{D}_\nu(\omega(e)) \psi_\rho + 2 \sqrt{2} \bar{\lambda} \Gamma^{\mu\nu} \mathcal{D}_\mu(\omega(e)) \psi_\nu \\
&\quad + 4 \bar{\lambda} \mathcal{D}(\omega(e)) \lambda + 3 \sqrt{2} \bar{\psi}_\mu \Gamma^\mu \Gamma^\nu \lambda (\phi^{-1} \partial_\nu \phi) - \frac{3}{2} \bar{\psi}_\mu \Gamma^\mu \psi_\nu (\phi^{-1} \partial^\nu \phi) + \frac{1}{16} \sqrt{2} H^{abc} C'_{abc} \\
&\quad + [1/(16 \times 24)] C'_{abc} C'_{abc} + \frac{1}{96} \bar{\psi}^\mu \Gamma^{abc} \psi_\mu (\bar{\lambda} \Gamma_{abc} \lambda + \frac{1}{2} \sqrt{2} \bar{\lambda} \Gamma_{abc} \Gamma \cdot \psi - \frac{1}{4} \bar{\psi}^\nu \Gamma_{abc} \psi_\nu - \frac{1}{8} \bar{\psi} \cdot \Gamma \Gamma_{abc} \Gamma \psi)],
\end{aligned} \tag{23}$$

$$\begin{aligned}
\mathcal{L}_\beta(F^2) &= e \phi^{-3} \beta \operatorname{tr} [-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} \bar{\chi} \not{D}(\omega(e, \psi), A) \chi - \frac{1}{8} \bar{\chi} \Gamma^\mu \Gamma^{ab} (F_{ab} + \hat{F}_{ab}) (\psi_\mu + \frac{1}{3} \sqrt{2} \Gamma_\mu \lambda) \\
&\quad - \frac{1}{16} \sqrt{2} \bar{\chi} \Gamma^{abc} \chi \hat{H}_{abc} + \frac{1}{96} \bar{\chi} \Gamma^{abc} \chi \bar{\lambda} \Gamma_{abc} \lambda - [1/(16 \times 24)] \sqrt{2} \bar{\chi} \Gamma^{abc} \chi \bar{\psi}_\mu (4 \Gamma_{abc} \Gamma^\mu + 3 \Gamma^\mu \Gamma_{abc}) \lambda] \\
&\quad - \frac{1}{4} i \beta \sqrt{2} \epsilon^{\alpha_1 \alpha_6 \mu\nu\lambda\rho} A_{\alpha_1 \alpha_6} \operatorname{tr}(F_{\mu\nu} F_{\lambda\rho}).
\end{aligned} \tag{24}$$

In (23) the tensor  $C'$  is the fermionic contribution to (20), i.e.,  $C' = -\frac{1}{2} C/A$ , with  $A$  and  $C$  as in (2). Note that  $\mathcal{L}_{\beta^2}$  has cancelled against the last term in (8). The action (23) + (24) is invariant under the following supersymmetry transformations:

$$\begin{aligned}
\delta_0 e_\mu^a &= \frac{1}{2} \bar{\epsilon} \Gamma^a \psi_\mu, \\
\delta_0 \psi_\mu &= (\partial_\mu - \frac{1}{4} \Omega_{\mu+}^{ab} \Gamma_{ab}) \epsilon + \frac{1}{2} \sqrt{2} (\epsilon \bar{\psi}_\mu \lambda - \psi_\mu \bar{\epsilon} \lambda + \Gamma^a \lambda \bar{\psi}_\mu \Gamma_a \epsilon),
\end{aligned} \tag{25}$$

$$\begin{aligned}
\delta_0 A_{\alpha_1 \alpha_6} &= [3/(4 \times 6!)] \sqrt{2} \phi^{-3} \bar{\epsilon} \Gamma_{[\alpha_1 \alpha_5} (\psi_{\alpha_6]} + \tfrac{1}{3} \sqrt{2} \Gamma_{\alpha_6} \lambda) , \\
\delta_0 \lambda &= -\tfrac{1}{8} \sqrt{2} \phi^{-1} \not{\phi} \phi \epsilon + \tfrac{1}{8} \Gamma^{abc} \epsilon (\hat{H}_{abc} - \tfrac{1}{24} \sqrt{2} \bar{\lambda} \Gamma_{abc} \lambda) , \\
\phi^{-1} \delta_0 \phi &= -\tfrac{1}{3} \sqrt{2} \bar{\epsilon} \lambda , \\
\delta_0 A_\mu &= \tfrac{1}{2} \bar{\epsilon} \Gamma_\mu \chi , \\
\delta_0 \chi &= -\tfrac{1}{4} \Gamma^{ab} \epsilon \hat{F}_{ab} + \tfrac{1}{2} \sqrt{2} (\epsilon \bar{\chi} \lambda - \chi \bar{\epsilon} \lambda + \Gamma^a \lambda \bar{\chi} \Gamma_a \epsilon) . \\
\delta_\beta \psi_\mu &= -\tfrac{1}{192} \beta \Gamma_\mu \Gamma^{abc} \epsilon \operatorname{tr}(\bar{\chi} \Gamma_{abc} \chi) , \\
\delta_\beta \lambda &= \tfrac{1}{128} \beta \sqrt{2} \Gamma^{abc} \epsilon \operatorname{tr}(\bar{\chi} \Gamma_{abc} \chi) .
\end{aligned} \tag{25 cont'd}$$

In these transformation rules we use (22) for  $H_{abc}$  and  $\hat{H}_{abc}$ , also in the definition (19) of  $\Omega_\pm$ . Note that the Yang–Mills action is now quadratic in the Yang–Mills fields (except for Yang–Mills covariantizations). This is a consequence of the superconformal origin of this action [8].

Let us now proceed with the duality transformation which includes the effect to  $O(\alpha)$  of the Lorentz Chern–Simons form and its supersymmetric partners. In ref. [3] we showed that in the two-form formulation the multiplet consisting of  $\Omega_{\mu-}^{ab}$  [see (19)] and  $\psi^{ab}$  #2 has precisely the transformation rules of an  $SO(9, 1)$  Yang–Mills multiplet. Therefore we can use (14) and (15) to obtain an invariant action for this multiplet, by replacing everywhere

$$A_\mu \rightarrow \Omega_{\mu-}^{ab}, \quad \chi \rightarrow \psi^{ab}, \quad \beta \rightarrow \alpha . \tag{27}$$

This must be done also in (13), because it contains the Yang–Mills fields in the Chern–Simons terms in  $H$ . The action is then invariant to  $O(\alpha)$  only, because the modifications (17) to the supersymmetry transformation rules (now  $\delta_\alpha$ ) cause new  $O(\alpha)$  supersymmetry transformations of  $\Omega_{\mu-}^{ab}$  and  $\psi^{ab}$ . The effect of these new transformations can then be cancelled by modifications of  $O(\alpha^2)$  to the transformation rules of the supergravity fields. The resulting iterative procedure was carried out to  $O(\alpha^3)$ . There is no essential problem in including a Yang–Mills sector throughout this procedure; the order reached this way ( $n=3$ ) contains  $\alpha^3$ ,  $\alpha^2\beta$ ,  $\alpha\beta^2$  contributions to the action and transformation rules.

The  $O(\alpha)$  action  $\mathcal{L}_\alpha(R^2)$  in the two-index formulation was given in eq. (B.28) of ref. [3]. The duality transformation at  $O(\alpha)$  is given by (7), with of course in addition the last term in (12), where  $X$  is now the Lorentz Chern–Simons form. We obtain for  $\mathcal{L}_\alpha(R^2)$

$$\begin{aligned}
\mathcal{L}_\alpha(R^2) &= e \phi^{-3} \alpha \{ -\tfrac{1}{4} R^{\mu\nu ab} (\Omega_-) R_{\mu\nu}{}^{ab} (\Omega_-) - \tfrac{1}{2} \bar{\psi}^{ab} \not{\phi} (\omega(e, \psi), \Omega_-) \psi_{ab} \\
&\quad - \tfrac{1}{8} \bar{\psi}^{ab} \Gamma^\mu \Gamma^{cd} [R_{cdab} (\Omega_-) + \hat{R}_{cdab} (\Omega_-)] (\psi_\mu + \tfrac{1}{3} \sqrt{2} \Gamma_\mu \lambda) + \tfrac{1}{16} \sqrt{2} \bar{\psi}^{de} \Gamma^{abc} \psi_{de} \hat{H}_{abc} \\
&\quad + \tfrac{1}{96} \bar{\psi}^{de} \Gamma^{abc} \psi_{de} \bar{\lambda} \Gamma_{abc} \lambda - [1/(16 \times 24)] \sqrt{2} \bar{\psi}^{de} \Gamma^{abc} \psi_{de} \bar{\psi}_\mu (4 \Gamma_{abc} \Gamma^\mu + 3 \Gamma^\mu \Gamma_{abc}) \lambda \} \\
&\quad - \tfrac{1}{4} i \alpha \sqrt{2} \epsilon_{\alpha_1 \alpha_6} \epsilon^{\mu\nu\lambda\rho} A_{\alpha_1 \alpha_6} R_{\mu\nu}{}^{ab} R(\Omega_-) R_{\lambda\rho ab} (\Omega_-) .
\end{aligned} \tag{28}$$

In this expression  $\hat{H}$  is given by (22). There is no difficulty in considering the combined actions  $\mathcal{L}_\alpha$  and  $\mathcal{L}_\beta$  with the corresponding transformation rules.

To find the supersymmetry transformation rules at  $O(\alpha, \beta)$  we must obtain the solution of (3) to  $O(\alpha, \beta)$ .  $t_0$  is given by (20), but instead of (21) we find

$$t_{1\mu\nu\rho} = \tfrac{1}{24} \sqrt{2} [\alpha \bar{\psi}^{ab} \Gamma_{\mu\nu\rho} \psi_{ab} + \beta \operatorname{tr}(\bar{\chi} \Gamma_{\mu\nu\rho} \chi)] - \alpha \sqrt{2} e^{-1} \phi^3 (e_{[\mu}{}^a e_\nu{}^b \mathcal{A}_{\rho]}{}^{ab} + \tfrac{1}{2} \bar{\psi}_a \Gamma_{[\mu\nu} e_{\rho]} b \mathcal{X}^{ab}) , \tag{29}$$

where

$$\mathcal{A}_\mu{}^{ab} \equiv \alpha^{-1} \delta \mathcal{L}_\alpha / \delta \Omega_\mu{}^{ab}, \quad \mathcal{X}^{ab} \equiv \alpha^{-1} \delta \mathcal{L}_\alpha / \delta \psi^{ab} . \tag{30}$$

#2  $\psi_{\mu\nu}$  is the supercovariant gravitino field strength:  $\psi_{\mu\nu} \equiv 2 \mathcal{D}_{[\mu} (\Omega_+) \psi_{\nu]} + \dots$

$\mathcal{A}_\mu^{ab}$  and  $\mathcal{X}^{ab}$  are the equations of motion of  $A_\mu$  and  $\chi$ , respectively, with the replacement (27). These two contributions, which were absent in (21), are due to the implicit dependence of  $\Omega_\pm$  on  $H$ . As we can see in (16), these two contributions cause additional  $O(\alpha)$  transformation rules of  $\psi_\mu$  and  $\lambda$ .

Let us now evaluate the transformation rule of  $A_{\alpha_1 \alpha_6}$  at  $O(\alpha)$ . We must therefore look for  $\partial_{[\lambda} t_{\mu\nu\rho]}$  in the variation of the first-order form of the action. The variation of the action which contributes is

$$\alpha(\mathcal{A}^{\mu ab}\delta_0\Omega_{\mu-}^{ab} + \mathcal{X}^{ab}\delta_0\psi_{ab}). \quad (31)$$

The fields  $\Omega_{\mu-}^{ab}$  and  $\psi^{ab}$  transform as  $SO(9, 1)$  Yang–Mills fields, but to establish this one has to use the Bianchi identity of  $H$  in the two-index formalism. In particular,

$$\delta_0\psi_{ab} = -\frac{1}{4}\Gamma^{cd}\epsilon R_{abcd}(\Omega_+) = -\frac{1}{4}\Gamma^{cd}\epsilon[R_{cdab}(\Omega_-) + 6\sqrt{2}\partial_{[a}t_{bcd]}], \quad (32)$$

where we have omitted fermionic terms. The  $\partial t$  term is of higher order in  $\alpha$  and  $\beta$  when  $t$  is replaced by the modified field-strength (11), but in this case it contributes to the  $O(\alpha)$  transformation of  $A_{\alpha_1 \alpha_6}$ . Including the Yang–Mills sector at  $O(\beta)$ , the  $O(\alpha, \beta)$  transformation rules of the supergravity fields are

$$\begin{aligned} (\delta_\alpha + \delta_\beta)\psi_\mu &= -\frac{1}{192}\Gamma_\mu\Gamma^{abc}\epsilon[\alpha\bar{\psi}^{de}\Gamma_{abc}\psi_{de} + \beta\text{tr}(\bar{\chi}\Gamma_{abc}\chi)] + \frac{3}{4}\alpha e^{-1}\phi^3\Gamma^{ab}\epsilon(\mathcal{A}_{[\mu ab]} + \frac{1}{2}\bar{\psi}_\lambda\Gamma_{[\mu a}\mathcal{X}^\lambda_{b]}), \\ (\delta_\alpha + \delta_\beta)\lambda &= +\frac{1}{128}\sqrt{2}\Gamma^{abc}\epsilon[\alpha\bar{\psi}^{de}\Gamma_{abc}\psi_{de} + \beta\text{tr}(\bar{\chi}\Gamma_{abc}\chi)] - \frac{1}{8}\alpha\sqrt{2}e^{-1}\phi^3\Gamma^{abc}\epsilon(\mathcal{A}_{abc} + \frac{1}{2}\bar{\psi}_\lambda\Gamma_{ab}\mathcal{X}^\lambda_c), \\ (\delta_\alpha + \delta_\beta)A_{\alpha_1 \alpha_6} &= [1/8(\times 6!)]\alpha\sqrt{2}\bar{\epsilon}\Gamma^{ab}_{\alpha_1 \alpha_6}\mathcal{X}_{ab}. \end{aligned} \quad (33)$$

This is in complete agreement with the results of ref. [1]. Note that we have improved on [1] by doing a complete analysis, including quartic fermions, of the action (28) and the corresponding transformation rules (33).

Let us now consider the duality transformation to  $O(\alpha^2)$ . For the action we have to use (8), and of course the last term in (12). The action  $\mathcal{L}_2$  (terms of  $O(\alpha^2, \alpha\beta, \beta^2)$ ) consists of the sum of

$$-[1/(16 \times 24)]e\phi^{-3}[\alpha\bar{\psi}^{ab}\Gamma^{\mu\nu\rho}\psi_{ab} + \beta\text{tr}(\bar{\chi}\Gamma^{\mu\nu\rho}\chi)]^2, \quad (34)$$

the analogue of (15) for this case, and of eq. (3.17) of ref. [3]. It indeed consists only of fermionic terms. Let us gather all terms at this order in the action *after* the duality transformation:

$$\begin{aligned} \mathcal{L}_2|_{t_0} - At_1^2 &= -\frac{9}{2}\alpha\sqrt{2}\partial_\lambda(e\phi^{-3}t_0^{\lambda\mu\nu})\bar{\psi}^\rho\Gamma_{[\rho}\chi_{\mu\nu]} - \frac{3}{2}\alpha\bar{\psi}_\lambda\Gamma^{ab}(\Psi_\rho + \frac{1}{4}\sqrt{2}\Gamma_\rho A)\Gamma^{\lambda\rho ab} \\ &\quad - \frac{1}{8}\alpha[\alpha\bar{\psi}^{ab}\Gamma^{\mu\nu\rho}\psi_{ab} + \beta\text{tr}(\bar{\chi}\Gamma^{\mu\nu\rho}\chi)](\mathcal{A}_{\mu\nu\rho} + \frac{1}{2}\bar{\psi}_\lambda\Gamma_{\mu\nu}\mathcal{X}^\lambda_\rho) + \frac{3}{2}\alpha^2e^{-1}\phi^3(\mathcal{A}_{[\mu\nu\rho]} + \frac{1}{2}\bar{\psi}_\lambda\Gamma_{[\mu\nu}\mathcal{X}^\lambda_{\rho]})^2. \end{aligned} \quad (35)$$

Here  $t_0$  and  $t_1$  are given in (20) and (29), respectively. Furthermore,  $\Psi^\mu$  and  $A$  are the  $O(\alpha^0)$  equations of motion of  $\psi_\mu$  and  $\lambda$ . Also, we have introduced the following tensors and tensor-spinors:

$$T_{\mu\nu\lambda\rho} = \alpha R_{[\mu\nu}{}^{ab}(\Omega_-)R_{\lambda\rho]}{}^{ab}(\Omega_-) + \beta\text{tr}(F_{[\mu\nu}F_{\lambda\rho]}),$$

$$T_{\mu\nu} = \alpha R_{\mu\lambda}{}^{ab}(\Omega_-)R^\lambda{}_\nu{}^{ab}(\Omega_-) + \beta\text{tr}(F_{\mu\lambda}F^\lambda{}_\nu),$$

$$T = T_\mu{}^\mu,$$

$$X_{\mu\nu} = \alpha R_{\mu\nu}{}^{ab}(\Omega_-)\psi^{ab} + \beta\text{tr}(F_{\mu\nu}\chi),$$

$$X_\mu = \Gamma^\nu X_{\mu\nu},$$

$$X = \Gamma^\mu X_\mu. \quad (36)$$

Note that in (35) the four-fermion contribution (34) has cancelled. In ref. [3] we did not consider four-fermion terms at this order, and therefore the first line of (35) is correct modulo terms quartic in fermions. We do not retain four-fermion terms from now on.

All terms in (35) can be eliminated by field redefinitions, or are quartic in fermions. The first term, containing  $t_0$ , is quartic in fermions since the leading contribution from (20) vanishes due to the Bianchi identity of

$R(A)$ . The second term we eliminate by redefining the gravitino  $\psi_\mu$  and  $\lambda$ , as follows:

$$\begin{aligned}\psi'_\mu &\equiv \psi_\mu - \frac{3}{2}\alpha\Gamma^{ab}\psi_\lambda T^\lambda_{\mu ab}, \\ \lambda' &= \lambda - \frac{1}{4}\sqrt{2}\Gamma^\mu(\psi'_\mu - \psi_\mu).\end{aligned}\quad (37)$$

The last two lines of (35) can also be defined away, because both  $\mathcal{A}_\mu^{ab}$  and  $\mathcal{X}^{ab}$  can be expressed in terms of  $O(\alpha^0)$  equations of motion of the supergravity fields. This was shown in ref. [3] for the two-index formulation, but, as pointed out in ref. [1], is equally valid for the dual version. For our present purposes we need only:

$$\mathcal{A}^{\mu ab} = e^\mu_c \{ 2e\phi^{-3}e^{\lambda a}e^{\rho b} \mathcal{D}_{[\lambda}[e^{-1}\phi^3(\mathcal{E}_{\rho]}^c + \frac{1}{3}e_{\rho]}^c\Phi)] - [1/(4 \times 6!)]i\sqrt{2}\mathcal{D}_\nu(e\phi^{-3}\epsilon^{\nu abc\alpha_1\alpha_6}\mathcal{A}_{\alpha_1\alpha_6}) \}, \quad (38)$$

where  $\mathcal{E}_a^\mu$ ,  $\Phi$  and  $\mathcal{A}_{\alpha_1\alpha_6}$  are the equations of motion of  $e_\mu^a$ ,  $\phi$  and  $A_{\alpha_1\alpha_6}$ , respectively.

Note that the last line of (35) is quadratic in equations of motion, and therefore the redefinition of the corresponding fields will be proportional to equations of motion. So even though this might give an  $O(\alpha^3, \alpha^2\beta, \alpha\beta^2)$  action, this can again be defined away to higher orders. For the second line of (35) we use (38) and redefine the zehnbein, the dilaton and the antisymmetric tensor gauge field (the contributions of  $\mathcal{X}^{ab}$  being quartic in fermions). The redefinition of  $A$  takes the form

$$A'_{\alpha_1\alpha_6} = A_{\alpha_1\alpha_6} + [1/(32 \times 6!)]i\alpha\sqrt{2}e\phi^{-3}\epsilon^{\nu abc}_{\alpha_1\alpha_6}\mathcal{D}_\nu[\alpha\bar{\psi}^{de}\Gamma_{abc}\psi_{de} + \beta\text{tr}(\bar{\chi}\Gamma_{abc}\chi)]. \quad (39)$$

After applying all redefinitions the cubic action is absent, as in ref. [1].

We finally consider the duality transformation at  $O(\alpha^3)$ . This is given in (10). Clearly all contributions containing  $t_1^2$  or  $t_1t_2$  are quartic in fermions or can be eliminated by redefinitions. Therefore the only contribution from (10) is  $\mathcal{L}_3|_{t_0}$ . However, the redefinitions (37) and (39) also contribute to the quartic action, when applied to the  $O(\alpha, \beta)$  action (24) and (28). Using the definitions (36), the sources of these contributions can be written:

$$\alpha e\phi^{-3}[\frac{1}{4}T - \frac{1}{4}\bar{X}_{ab}\Gamma^\mu T^{ab}(\psi_\mu + \frac{1}{3}\sqrt{2}\Gamma_\mu\lambda)] - \frac{1}{4}i\sqrt{2}\epsilon^{\alpha_1\alpha_6\mu\nu\lambda\rho}A_{\alpha_1\alpha_6}T_{\mu\nu\lambda\rho}. \quad (40)$$

Applying (37) and (39) to (40), we find the following new terms in the quartic action:

$$\frac{3}{2}\alpha e\phi^{-3}\bar{\psi}_\mu\Gamma_{\nu\lambda}X_\rho T^{\mu\nu\lambda\rho} - \frac{3}{8}\alpha[\alpha\bar{\psi}^{de}\Gamma_{abc}\psi_{de} + \beta\text{tr}(\bar{\chi}\Gamma_{abc}\chi)]\mathcal{D}_\nu(e\phi^{-3}T^{\nu abc}). \quad (41)$$

The redefinition of the zehnbein derived from (35) is antisymmetric, and does not contribute when applied to (40). In ref. [1] we obtained, using the Noether procedure, exactly the same difference (41) between the quartic actions in the two formulations of  $d=10$  supergravity.

In summary, we have seen that the quartic effective actions in the dual formulations of  $d=10$ ,  $N=1$  supergravity have the same structure. Essentially, one can be obtained from the other by choosing  $t_0$  either as the field strength of the two-index gauge field modified with Chern–Simons forms (11), or as the dual of the field strength of  $A_{\alpha_1\alpha_6}$  (20). The role of the Chern–Simons forms is then taken over by the last term in (12). The slight differences in the fermionic terms of the quartic action obtained in ref. [1] can be understood from the field redefinitions mentioned above. We expect that at  $O(\alpha^6)$  the bosonic part of the actions for the two formulations will also be different.

Note that the fact that the duality transformation arises from changing the interpretation of  $t$  in a first-order formulation is similar to the duality transformation in superspace discussed in ref. [10] at  $O(\alpha)$ . Duality transformations which include in particular the counterterms which make  $d=10$ ,  $N=1$  supergravity coupled to Yang–Mills anomaly-free, were considered in ref. [11].

In this letter we have converted the two-index form to a six-index form in the heterotic string effective action by an iterative duality transformation. The two formulations of the effective action are equivalent in the sense that both can be derived iteratively from the same first-order formulation. The six-index form couples in a natural way to five-branes rather than to strings. Five-branes as classical solutions of ten-dimensional Yang–Mills theory [12], and of the  $d=10$  heterotic string effective action [13] have been considered recently. It would



be interesting to see whether the dynamics of the zero-modes of such five-branes can be described by a Green–Schwarz type action together with heterotic gauge fermions defined on a six-dimensional world sheet. This is suggested by the work of ref. [13]. The requirement that such an action is invariant under  $\kappa$ -symmetries should then yield the superspace constraints corresponding to the dual formulation of  $d=10$  supergravity coupled to Yang–Mills.

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